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A new modified King–Werner method for solving nonlinear equations ☆

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ABSTRACT

In this paper, a new method for solving nonlinear equations $f(x) = 0$ is presented. Analysis of the convergence shows that the asymptotic convergence order of this method is $1 + \sqrt{3}$. Some numerical results are given to demonstrate its efficiency.

1. Introduction

Numerical methods for solving nonlinear equations are a popular and important research topic in numerical analysis. In this paper, we consider iterative methods to find a simple root of a nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$, for an open interval D, is a scalar function.

We all know that Newton's method is an important and basic approach for solving nonlinear equations [1–5], and its formulation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

This method is a quadratic method.

To improve the local order of convergence, a number of modified methods have been studied and reported in the literature [6–14]. By employing a second derivative evaluation we can obtain some well-known third-order methods, such as Chebyshev's method, Halley's method [6] and the super-Halley method [8]. However, in many other cases, it is expensive to compute the derivative, and the above methods are still restricted in practical applications. The well known secant method is given by

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n). \tag{2}$$

The method does not require any derivative, but its order is only 1.618. To improve this method, many modified methods called the secant-like method have been proposed in [9,15,16].

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Recently, Zhang et al. [15] and Wang et al. [16] have proposed a new two-step secant-like method and a new three-step secant-like method respectively, and the convergence order of these methods are improved. Zhang's method proposed in [15] is very similar to the well-known King–Werner method with order 2.414 [17], which has the form [18,19]:

$$\begin{cases} x_{n+1} = x_n - f'\left(\dfrac{x_n + y_n}{2}\right)^{-1} f(x_n), \\ y_{n+1} = x_{n+1} - f'\left(\dfrac{x_n + y_n}{2}\right)^{-1} f(x_{n+1}). \end{cases} \tag{3}$$

Here, we attempt to improve the order of the classical method defined by (1), and (2) by using previous information, and then we present a new iterative method for solving nonlinear equations. Analysis of the convergence shows that the asymptotic convergence order of this method is $1 + \sqrt{3}$ which is equal to the method defined by Wang et al. proposed in [16]. Per iteration, the methods require two evaluations of the function and one of its first derivative and therefore the efficiency, in term of function evaluations, of the new methods is equal to $\sqrt[3]{1.618} \approx 1.17398$, which is better than the method proposed in [16]. Finally, some numerical examples are also given.

2. Notation and basic results

Let $f(x)$ be a real function with a simple root x^* and let $\{x_n\}_{n=0}^{\infty}$ be a sequence of real numbers that converges to x^*. We say that the *order of convergence* is q if there exists a $q \in \mathbb{R}^+$ such that

$$\lim_{n \to +\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^q} = C \neq 0, \infty.$$

Let $e_n = x_n - x^*$ be the n-th iterate error. We call

$$e_{n+1} = C e_n^q + \cdots, \tag{4}$$

the *error equation*. If we can obtain the error equation for the method, then the value of q is its order of convergence.

3. The new method

Here, in order to construct our method, we use the following second-order polynomial function proposed in [16]:

$$\begin{aligned} P(x) &= f(x_n) + v_n^{-1}(x - x_n) \\ &\quad + \frac{(v_{n-1}^{-1} - v_n^{-1})(x - x_n)(x - y_n)}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n} \end{aligned} \tag{5}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and $y_n = x_n - v_{n-1} f(x_n)$, $v_n = f'\left(\frac{x_n + y_n}{2}\right)^{-1}$ and $z_n = x_n - v_n f(x_n)$, respectively.

It is easy to obtain that

$$(v_{n-1}^{-1} - v_n^{-1})(y_n - x_n)(z_n - y_n) = v_n^{-1}(y_n - z_n)^2.$$

In order to eliminate the nonlinearity, we replace x_{n+1} in the terms $(x_{n+1} - x_n)$ and $(x_{n+1} - y_n)$ of $P(x_{n+1})$ with y_n and z_n, respectively. By solving the function (5), we can obtain a new method

$$\begin{cases} y_n = x_n - v_{n-1} f(x_n), \\ v_n = f'\left(\dfrac{x_n + y_n}{2}\right)^{-1} \\ z_n = x_n - v_n f(x_n), \\ x_{n+1} = z_n - \dfrac{(y_n - z_n)^2}{\alpha_1 x_{n-1} + \alpha_2 y_{n-1} + (2 - \alpha_1 - \alpha_2) z_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n} \end{cases} \tag{6}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$.

The method defined by (6) can be viewed as an iterative method with three substeps. The first two substeps are the well-known King–Werner method [18,19]. The third substep is an acceleration by using the values computed previously.

At the beginning of the process, the values of x_0, y_0 and v_{-1} need to be given by some approaches. The choice of v_{-1} cannot affect the asymptotic convergence order of the method defined by (6) while $v_{-1} \neq 0$. Since the first iteration cannot carry out the third substep, and hence we let $x_1 = z_0$.

Theorem 1. *Assume that the function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D has a simple root $x^* \in D$. Let $f(x)$ have first, second and third derivatives in the interval D, then the asymptotic convergence of the method defined by (6) is $1 + \sqrt{3}$ when $\alpha_1 = \alpha_2 = 1$.*

Proof. Let $d_n = y_n - x^*$ and $w_n = z_n - x^*$. Using Taylor expansion, we get

$$f'\left(\frac{x_n + y_n}{2}\right) = f'(x^*)\left[1 + C_2(e_n + d_n) + \frac{3}{4}C_3(e_n + d_n)^2 + \frac{1}{2}C_4(e_n + d_n)^3 \cdots\right] \tag{7}$$

where $C_k = \frac{1}{k!}\frac{f^{(k)}(x^*)}{f'(x^*)}$, $k = 2, 3, \ldots$. we have

$$\begin{aligned} v_n &= f'\left(\frac{x_n + y_n}{2}\right)^{-1} \\ &= \frac{1}{f'(x^*)}\frac{1}{1 + C_2(e_n + d_n) + \frac{3}{4}C_3(e_n + d_n)^2 + \frac{1}{2}C_4(e_n + d_n)^3 + \cdots} \\ &= \frac{1}{f'(x^*)}\left[1 - C_2(e_n + d_n) - \left(\frac{3}{4}C_3 - C_2^2\right)(e_n + d_n)^2 + \cdots\right]. \end{aligned} \tag{8}$$

Furthermore, we have

$$f(x_n) = f'(x^*)[e_n + C_2e_n^2 + C_3e_n^3 + \cdots]. \tag{9}$$

Thus it follows (7), (8), (9) and $y_n = x_n - v_{n-1}f(x_n)$ that

$$\begin{aligned} d_n &= x_n - x^* - v_{n-1}f(x_n) \\ &= e_n - \left[1 - C_2(e_{n-1} + d_{n-1}) - \left(\frac{3}{4}C_3 - C_2^2\right)(e_{n-1} + d_{n-1})^2 + \cdots\right][e_n + C_2e_n^2 + C_3e_n^3 + \cdots] \\ &= C_2e_n(e_{n-1} + d_{n-1}) + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1} + d_{n-1})^2 - C_2e_n^2 + C_2^2e_n^2(e_{n-1} + d_{n-1}) + \cdots \end{aligned} \tag{10}$$

and hence, we obtain

$$\begin{aligned} w_n &= x_n - x^* - v_nf(x_n) \\ &= e_n - \left[1 - C_2(e_n + d_n) - \left(\frac{3}{4}C_3 - C_2^2\right)(e_n + d_n)^2 + \cdots\right][e_n + C_2e_n^2 + C_3e_n^3 + \cdots] \\ &= C_2e_nd_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_n + d_n)^2 - C_2^2e_n^2(e_n + d_n) + \cdots \\ &= C_2e_nd_n + \left[\frac{3}{4}C_3e_n + \left(\frac{3}{4}C_3 - C_2^2\right)d_n\right]e_n(e_n + d_n) + \cdots. \end{aligned} \tag{11}$$

From (6), we obtain

$$e_{n+1} = w_n - \frac{(d_n - w_n)^2}{\alpha_1e_{n-1} + \alpha_2d_{n-1} + (2 - \alpha_1 - \alpha_2)w_{n-1} - \beta_1e_n - \beta_2d_n - (2 - \beta_1 - \beta_2)w_n}. \tag{12}$$

We first consider the case $\alpha_1 = 0, \alpha_2 \neq 0$, and in this case we have

$$\begin{aligned} e_{n+1} &= C_2e_nd_n - \frac{1}{\alpha_2}\frac{C_2e_n\left(\frac{e_{n-1}}{d_{n-1}} + 1\right) + \cdots}{1 + \frac{2-\alpha_2}{\alpha_2}\frac{w_{n-1}}{d_{n-1}} + \cdots}(d_n - w_n) + \cdots \\ &= C_2e_nd_n - \frac{e_nd_n}{\alpha_2e_{n-2}} + \cdots = -\frac{C_2e_n^2e_{n-1}}{\alpha_2e_{n-2}} + \cdots. \end{aligned} \tag{13}$$

From (11)–(13), we can see that, from z_n to x_{n+1}, the order is not improved when $\alpha_1 = 0, \alpha_2 \neq 0$. Similarly, it is obtained that the case $\alpha_1 = \alpha_2 = 0$ also cannot improve the order. We now turn to consider the case $\alpha \neq 0$, and using (10)–(12), we obtain

$$\begin{aligned} e_{n+1} &= C_2e_nd_n - \frac{1}{\alpha_1}\frac{C_2e_n\left(\frac{d_{n-1}}{e_{n-1}} + 1\right) + \cdots}{1 + \frac{\alpha_2}{\alpha_1}\frac{d_{n-1}}{e_{n-1}} + \cdots}(d_n - w_n) + \cdots \\ &= \left(1 - \frac{1}{\alpha_1}\right)C_2e_nd_n - \frac{1}{\alpha_1}\left(1 - \frac{\alpha_2}{\alpha_1}\right)C_2e_nd_n\frac{d_{n-1}}{e_{n-1}} + \cdots \end{aligned}$$

$$
\begin{aligned}
&= \left(1-\frac{1}{\alpha_1}\right)C_2e_nd_n - \frac{1}{\alpha_1}\left(1-\frac{\alpha_2}{\alpha_1}\right)C_2e_nd_ne_{n-2} + \cdots \\
&= \left(1-\frac{1}{\alpha_1}\right)C_2e_nd_n + \left[1-\frac{1}{\alpha_1}\left(2-\frac{\alpha_2}{\alpha_1}\right)\right]C_2^3e_n^2e_{n-1}e_{n-2} + \cdots. \qquad (14)
\end{aligned}
$$

From (14), we can see that the order will be improved by taking $\alpha_1 = \alpha_2 = 1$. In the following, by letting $\alpha_1 = \alpha_2 = 1$, and from (10)–(12), we obtain

$$
\begin{aligned}
e_{n+1} &= w_n - \frac{d_n - w_n}{e_{n-1}+d_{n-1}-\beta_1e_n-\beta_2d_n-(2-\beta_1-\beta_2)w_n}(d_n-w_n) \\
&= C_2e_nd_n - (d_n-w_n)\frac{C_2e_n(e_{n-1}+d_{n-1}) + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1}+d_{n-1})^2 + \cdots}{e_{n-1}+d_{n-1}-\beta_1e_n-\beta_2d_n-(2-\beta_1-\beta_2)w_n} + \cdots \\
&= C_2e_nd_n - \frac{C_2e_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1}+d_{n-1}) + \cdots}{1-\frac{\beta_1e_n-\beta_2d_n-(2-\beta_1-\beta_2)w_n}{e_{n-1}+d_{n-1}}}(d_n-w_n) + \cdots \\
&= C_2e_nd_n - \left[C_2e_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1}+d_{n-1}) + \cdots\right] \\
&\quad \times \left[1+\frac{\beta_1e_n-\beta_2d_n-(2-\beta_1-\beta_2)w_n}{e_{n-1}+d_{n-1}} + \cdots\right](d_n-w_n) + \cdots \\
&= C_2e_nd_n - \left[C_2e_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1}+d_{n-1}) + \cdots\right]\left[1+\frac{\beta_1e_n+\cdots}{e_{n-1}+d_{n-1}} + \cdots\right](d_n-w_n) + \cdots \\
&= C_2e_nd_n - \left[C_2e_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1}+d_{n-1}) + \cdots\right]\left[1+\frac{\beta_1\frac{e_n}{e_{n-1}}}{1+\frac{d_{n-1}}{e_{n-1}}} + \cdots\right](d_n-w_n) + \cdots \\
&= C_2e_nd_n - \left[C_2e_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_n(e_{n-1}+d_{n-1}) + \cdots\right]\left[1+\beta_1\frac{e_n}{e_{n-1}}\left(1-\frac{d_{n-1}}{e_{n-1}}\right) + \cdots\right](d_n-w_n) + \cdots \\
&= C_2e_nd_n - \left[C_2e_n + \left(\frac{3}{4}C_3 - C_2^2\right)e_ne_{n-1} + C_2\beta_1\frac{e_n^2}{e_{n-1}} + \cdots\right](d_n-w_n) + \cdots \\
&= \left(C_2^2 - \frac{3}{4}C_3\right)e_ne_{n-1}d_n - C_2\beta_1\frac{e_n^2}{e_{n-1}}d_n + \cdots \\
&= C_2\left(C_2^2 - \frac{3}{4}C_3\right)e_n^2e_{n-1}^2 - C_2^2\beta_1e_n^3 + \cdots \\
&= C_2\left(C_2^2 - \frac{3}{4}C_3\right)e_n^2e_{n-1}^2 + \cdots. \qquad (15)
\end{aligned}
$$

Let $A = C_2\left(C_2^2 - \frac{3}{4}C_3\right)$, then (15) becomes

$$e_{n+1} = Ae_n^2e_{n-1}^2 + \cdots. \qquad (16)$$

Suppose that the order of (6) is q when $\alpha_1 = \alpha_2 = 1$, then from (4) we have

$$e_n = Ce_{n-1}^q + \cdots \qquad (17)$$

and

$$e_{n+1} = Ce_n^q + \cdots = C^{q+1}e_{n-1}^{q^2} + \cdots. \qquad (18)$$

Substituting (17) and (18) into (16) gives

$$C^{q+1}e_{n-1}^{q^2} = AC^2e_{n-1}^{2q+2} + \cdots \qquad (19)$$

which implies that

$$q^2 = 2q + 2. \qquad (20)$$

It is obtained from (20) that the asymptotic convergence order $q = 1 + \sqrt{3}$. $\square$

Table 1
Comparison of various iterative methods for Example 1.

	Secant method	Newton's method	New method
	$x_{-1} = -2, x_0 = -1$	$x_0 = -1$	$x_0 = -1$
n	8	5	4
$\lvert x_n - x_{n-1}\rvert$	3.34199334872665e−012	2.22044604925031e−016	2.22044604925031e−016
$\lvert f(x_n)\rvert$	3.5527136788005e−015	2.66453525910038e−015	2.66453525910038e−015

Table 2
Comparison of various iterative methods for Example 2.

	Secant method	Newton's method	New method
	$x_{-1} = 2, x_0 = 1.5$	$x_0 = 1.5$	$x_0 = 1.5$
n	6	4	3
$\lvert x_n - x_{n-1}\rvert$	2.44204656496549e−012	2.22044604925031e−016	2.22044604925031e−016
$\lvert f(x_n)\rvert$	3.33066907387547e−016	4.44089209850063e−016	4.44089209850063e−016

Table 3
Comparison of various iterative methods for Example 3.

	Secant method	Newton's method	New method
	$x_{-1} = 3.05, x_0 = 2.98$	$x_0 = 2.98$	$x_0 = 2.98$
n	7	4	3
$\lvert x_n - x_{n-1}\rvert$	3.29958282918597e−013	2.22044604925031e−015	0
$\lvert f(x_n)\rvert$	0	0	0

By Theorem 1, we take $\alpha_1 = \alpha_2 = 1$ in (6), and obtain the present method given by

$$\begin{cases} y_n = x_n - v_{n-1} f(x_n), \\ v_n = f'\left(\dfrac{x_n + y_n}{2}\right)^{-1} \\ z_n = x_n - v_n f(x_n), \\ x_{n+1} = z_n - \dfrac{(y_n - z_n)^2}{x_{n-1} + y_{n-1} - \beta_1 x_n - \beta_2 y_n - (2 - \beta_1 - \beta_2) z_n} \end{cases} \tag{21}$$

where $\beta_1, \beta_2 \in \mathbb{R}$.

Per iteration the present methods require two evaluations of the function and one of its first derivative. We consider the definition of efficiency index [5] as $p^{1/\omega}$, where p is the order of the method and ω is the number of function evaluations per iteration required by the method. We have that the present methods have the efficiency index equal to $\sqrt[3]{1.618} \approx 1.17398$, which is better than the method proposed in [16] $\sqrt[5]{1.618} \approx 1.10102$.

4. Numerical example

Now we employ the new method given by (21) with $\beta_1 = \beta_2 = 0$ and $v_{-1} = 1$ to solve some nonlinear equations and compare them with the secant method and Newton's method defined by (1) proposed in [1–4]. The iterative method is stopped when $|f(x)| < 1e - 14$ or $|x_n - x_{n-1}| < 1e - 14$. The examples are as follows.

Example 1.

$$f(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \qquad x^* = -1.20764782713092.$$

The results of this problem are displayed in Table 1.

Example 2.

$$f(x) = \sin^2 x - x^2 + 1, \qquad x^* = 1.40449164821534.$$

The results of this problem are displayed in Table 2.

Example 3.

$$f(x) = e^{x^2+7x-30} - 1, \qquad x^* = 3.$$

The results of this problem are displayed in Table 3.

From the tables, we see that the new method is efficient. It converges faster than not only the secant method but also Newton's method. In view of this fact, the new method can be viewed as a significant improvement compared with the previously known methods.

5. Conclusions

We present a new iterative method for solving nonlinear equations. Theorem 1 shows that the asymptotic convergence order of this method is $1+\sqrt{3}$. The numerical example shows that the method is efficient.

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